Equation with Fibonacci coefficients.

https://www.linkedin.com/feed/update/urn:li:activity:6746091436783001600

Solve for $x, y \in \mathbb{R}$

$$\frac{F_{2022} \cdot x + F_{2021} \cdot y}{x + y} = \varphi^{2020}$$

where F_n is n-th Fibonacci's number and $\varphi = \dfrac{1 + \sqrt{5}}{2}$ (the golden's number).

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Since $\varphi^{n+1} - \varphi^n - \varphi^{n-1} = 0, \forall n \in \mathbb{N}$ (because $\varphi^2 - \varphi - 1 = 0$) there are unique $c_1, c_2 \in \mathbb{R}$ such that $\varphi^n = c_1 F_n + c_2 F_{n+1}, n \in \mathbb{N} \cup \{0\}$.

Indeed, $\varphi^0 = 1 = c_1 F_0 + c_2 F_1 \Leftrightarrow c_2 = 1$, $\varphi^1 = c_1 F_1 + c_2 F_2 \Leftrightarrow c_1 = \varphi - 1$ and for any $n \in \mathbb{N}$ assuming $\varphi^n = (\varphi - 1)F_n + F_{n+1}, \varphi^{n-1} = (\varphi - 1)F_{n-1} + F_n$ we have $\varphi^{n+1} = \varphi^n + \varphi^{n-1} = (\varphi - 1)F_{n-1} + F_n + (\varphi - 1)F_n + F_{n+1} = (\varphi - 1)(F_{n-1} + F_n) + (\varphi - 1)(F_n + F_{n+1}) = (\varphi - 1)F_{n+1} + F_{n+2}$.

Thus, by Math Induction, $\varphi^n = (\varphi - 1)F_n + F_{n+1}, \forall n \in \mathbb{N} \cup \{0\}$ and, therefore, for any $n \in \mathbb{N}$ we have $\varphi^{n-1} = \dfrac{\varphi - 1}{\varphi} \cdot F_n + \dfrac{1}{\varphi} \cdot F_{n+1} = \dfrac{F_{n+1} \cdot x + F_n \cdot y}{x + y}$, where $x = \dfrac{1}{\varphi}, y = \dfrac{\varphi - 1}{\varphi}$ and, therefore, $\dfrac{x}{y} = \dfrac{1}{\varphi - 1} = \varphi$.

In particular $\dfrac{F_{2022} \cdot x + F_{2021} \cdot y}{x + y} = \varphi^{2020} \Leftrightarrow x = \dfrac{1}{\varphi}, y = \dfrac{\varphi - 1}{\varphi}$.